TORIC IDEALS OF MATCHING POLYTOPES AND GRAPH COLORING THEORY

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This talk is based on joint work with Ryo Motomura, Hidefumi Ohsugi, and Akiyoshi Tsuchiya [6].

In this talk, we discuss a relationship between an algebraic property of toric ideals arising from graphs and a combinatorial property of edge colorings of multigraphs. Throughout this talk, we assume that a graph is simple, namely, it has no loops and no multiple edges and, a multigraph has no loops. Let G be a graph with the vertex set $V(G) = [d] := \{1,2,\dots,d\}$ and the edge set $E(G) = \{e_1,\dots,e_n\}$. A *matching* of G is a set of pairwise non-adjacent edges of G, and a *perfect matching* of G is a matching that covers every vertex of G. Let $M(G)$ (resp. $PM(G)$) denote the set of all matchings (resp. perfect matchings) of G. Given a subset $M \subset E(G)$, we associate the $(0,1)$-vector $\rho(M) = \sum_{e_j \in M} \mathbf{e}_j \in \mathbb{R}^n$. Here $\mathbf{e}_j$ is the jth unit coordinate vector in $\mathbb{R}^n$. For example, $\rho(\emptyset) = (0,\dots,0) \in \mathbb{R}^n$. Then the (*full*) *matching polytope* $\mathscr{M}_G$ of G is defined as the convex hull

$$\mathscr{M}_G = \mathrm{conv}\{\rho(M) : M \in M(G)\}$$

and the *perfect matching polytope* $\mathscr{P}_G$ of G is defined as

$$\mathscr{P}_G = \mathrm{conv}\{\rho(M) : M \in PM(G)\}.$$

Note that $\mathscr{P}_G$ is a face of $\mathscr{M}_G$. Moreover, the perfect matching polytope of a complete bipartite graph $K_{d,d}$ is called the *Birkhoff polytope*, denoted by $\mathscr{B}_d$.

Next, we introduce toric rings and toric ideals. Let $\mathscr{P} \subset \mathbb{R}^d_{\geq 0}$ be a lattice polytope with $\mathscr{P} \cap \mathbb{Z}^d_{\geq 0} = \{\mathbf{a}_1,\dots,\mathbf{a}_n\}$ and let $\mathbb{K}[\mathbf{t},s] := \mathbb{K}[t_1,\dots,t_d,s]$ be the polynomial ring in $d+1$ variables over a field $\mathbb{K}$. Given a nonnegative integer vector $\mathbf{a} = (a_1,\dots,a_d) \in \mathbb{Z}^d_{\geq 0}$, we write $\mathbf{t}^{\mathbf{a}} := t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d} \in \mathbb{K}[\mathbf{t},s]$. The *toric ring* of $\mathscr{P}$ is

$$\mathbb{K}[\mathscr{P}] := \mathbb{K}[\mathbf{t}^{\mathbf{a}_1}s,\dots,\mathbf{t}^{\mathbf{a}_n}s] \subset \mathbb{K}[\mathbf{t},s].$$

We regard $\mathbb{K}[\mathscr{P}]$ as a homogeneous algebra by setting each $\deg(\mathbf{t}^{\mathbf{a}_i}s) = 1$. Let $R[\mathscr{P}] = \mathbb{K}[x_1,\dots,x_n]$ denote the polynomial ring in n variables over $\mathbb{K}$ with each $\deg(x_i) = 1$. The *toric ideal* $I_{\mathscr{P}}$ of $\mathscr{P}$ is the kernel of the surjective homomorphism $\pi : R[\mathscr{P}] \to \mathbb{K}[\mathscr{P}]$ defined by $\pi(x_i) = \mathbf{t}^{\mathbf{a}_i}s$ for $1 \leq i \leq n$. Note that $I_{\mathscr{P}}$ is a prime ideal generated by homogeneous binomials. The toric ring $\mathbb{K}[\mathscr{P}]$ is called *quadratic* if $I_{\mathscr{P}}$ is generated by quadratic binomials. For a homogeneous ideal I, let $\omega(I)$ denote the maximal degree of minimal generators of I. We say that "$I_{\mathscr{P}}$ is generated by quadratic binomials" even if $I_{\mathscr{P}} = \{0\}$. In particular, $\omega(I_{\mathscr{P}}) \geq 2$ and $\omega(\{0\}) = 2$.

In [4], it was conjectured that the toric ideal $I_{\mathscr{B}_n}$ of the Birkhoff polytope $\mathscr{B}_n$ is generated by binomials of degree at most 3, and this conjecture was shown in [7]. Moreover, in [5], by using this result, the toric ideal of a flow polytope is generated by binomials of degree at most 3. For a homogeneous ideal I, let $\omega(I)$ denote the maximal degree of minimal generators of I. Since the matching polytope of a bipartite graph is unimodularly equivalent to a flow polytope, the following result holds:

Theorem 1 ([5]). *For a bipartite graph G, one has $\omega(I_{\mathscr{M}_G}) \leq 3$.*

Next, we recall edge-colorings of multigraphs. Let G be a multigraph. For a k-edge-coloring f of G and a color $1 \leq j \leq k$, let $M^{(e)}(f, j)$ denote the set of all edges of color j. We say that two k-edge-colorings f and g of G *differ by an m-colored subgraph* if there is a set of colors S of size m such that $M^{(e)}(f, j) \neq M^{(e)}(g, j)$ for each $j \in S$, but $M^{(e)}(f, j) = M^{(e)}(g, j)$ for each $j \notin S$. For two k-edge-colorings f, g of G, we write $f \sim_r g$ if there exists a sequence $f_0, f_1, \ldots, f_s$ of k-edge-colorings of G with $f_0 = f$ and $f_s = g$ such that f_i differs from f_{i-1} by a k_i-colored subgraph with $k_i \leq r$. Note that $f \sim_r g$ implies $f \sim_{r+1} g$.

In [1, 2, 3], the following result was shown:

Theorem 2 ([1, 2, 3])**.** *Let G be a bipartite multigraph. Then for any k-edge-colorings f and g of G, one has $f \sim_3 g$.*

For a simple graph G on $[d]$ with $E(G) = \{e_1, e_2, \ldots, e_n\}$ and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$, let $G^{(e)}_{\mathbf{a}}$ be the multigraph on $[d]$ such that $G^{(e)}_{\mathbf{a}}$ has a_i multiedges e_i for each i. We call $G^{(e)}_{\mathbf{a}}$ the *edge-replication multigraph* of G on $\mathbf{a}$. Then our main result is the following:

Theorem 3. *Let G be a graph with n edges. Then $\omega(I_{\mathscr{M}_G}) \leq r$ if and only if for any $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$ and for any k-edge-colorings f and g of $G^{(e)}_{\mathbf{a}}$, one has $f \sim_r g$.*

Since any edge-replication multigraph of a simple bipartite graph is bipartite, Theorems 1 and 2 are equivalent from this theorem.

On the other hand, we give a characterization of a bipartite graph such that $\omega(I_{\mathscr{M}_G}) = 2$, i.e., $I_{\mathscr{M}_G}$ is generated by quadratic binomials. In fact,

Theorem 4. *Let G be a bipartite graph. Then the following conditions are equivalent*:

(i) $\omega(I_{\mathscr{M}_G}) = 2$;

(ii) *G has no odd subdivision of $K_{2,3}$ as a subgraph*;

(iii) *each block of G is a bipartite graph having no odd subdivision of $K_{2,3}$ as a subgraph.*

Otherwise, one has $\omega(I_{\mathscr{M}_G}) = 3$.

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